MATH2230 Complex Variables with Application
Suggested Solution for HW7

Sect. 46 No. 1

(a) $\int_C f(z)\,dz = \int_0^\pi \frac{2e^{i\theta}+2}{2e^{i\theta}}\, 2ie^{i\theta}\, d\theta = \int_0^\pi 2(e^{i\theta}+1)\, i\, d\theta$

$= 2i\left[\frac{e^{i\theta}}{i}+\theta\right]\Big|_0^\pi = 2\pi i - 4$

(b) $\int_C f(z)\,dz = 2i\left[\frac{e^{i\theta}}{i}+\theta\right]\Big|_\pi^{2\pi} = 2\pi i + 4$

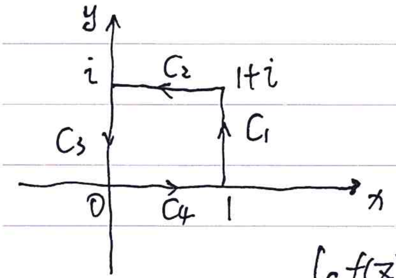
(c) $\int_C f(z)\,dz = 4\pi i$ $(C = (b) + (a))$

Sect. 46 No. 2

(a) $\int_C f(z)\,dz = \int_\pi^{2\pi} e^{i\theta}\, ie^{i\theta}\, d\theta = \int_\pi^{2\pi} ie^{2i\theta}\, d\theta = i\left[\frac{e^{2i\theta}}{2i}\right]_\pi^{2\pi} = 0$

(b) $\int_C f(z)\,dz = \int_0^2 (x-1)\,dx = \left[\frac{x^2}{2}-x\right]_0^2 = 0$

Sect. 46 No. 3.



$C_1: z = 1+ti \quad 0 \le t \le 1$

$C_2: z = (1-t)+i \quad 0 \le t \le 1$

$C_3: z = (1-t)i \quad 0 \le t \le 1$

$C_4: z = t \quad 0 \le t \le 1$

$\int_C f(z)\,dz = \sum_{k=1}^4 \int_{C_k} f(z)\,dz = 4(e^\pi - 1)$

(For details, refer to tutorial notes.)

Sect. 46 No. 4

Solution: Since C is the arc from $z = -1-i$ to $z = 1+i$ along $y = x^3$

we have $C: \; z = x+ix^3 \quad -1 \le x \le 1$

$\int_C f(z)\,dz = \int_{-1}^0 (1+3x^2 i)\,dx + \int_0^1 4x^3(1+3x^2 i)\,dx$

$= [x+ix^3]\Big|_{-1}^0 + [x^4+2x^6 i]\Big|_0^1 = 2+3i$

Sect. 47 No. 1

(a) proof: Noted that $|z+4| \le 6$ and $|z^3-1| \ge ||z|^3-1| = 7$

we have $\left|\frac{z+4}{z^3-1}\right| \le \frac{6}{7}$

Thus, $\left|\int_C \frac{z+4}{z^3-1}\,dz\right| \le \frac{6}{7}\pi$

(b) proof: $|z^2-1| \ge ||z|^2-1| = 3$

Thus, $\left|\int_C \frac{dz}{z^2-1}\right| \le \frac{\pi}{3}$

Sect. 47 No. 4

proof: $|2z^2-1| \le 2|z|^2+1 = 2R^2+1$

$|z^4+5z^2+4| = |z^2+1| \cdot |z^2+4|$

$|z^2+1| \ge ||z|^2-1| = R^2-1 \quad (R>2)$

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$$|z^2+4| \geq ||z|^2-4| = R^2-4 \quad (R>2)$$

And since the length of the upper half of the circle is πR, we have

$$\left|\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}dz\right| \leq \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}$$

Obviously, as $R\to\infty$, we have the integral tends to 0.

Sect. 49 No. 2

(a) $\int_0^{1+i} z^2 dz = [\frac{1}{3}z^3]\Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = 1/3 + i + i^2 - \frac{1}{3}i = \frac{2}{3}(-1+i)$

(b) $\int_0^{\pi+2i} \cos(\frac{z}{2})dz = [2\sin\frac{z}{2}]\Big|_0^{\pi+2i} = \left[\frac{e^{i\frac{z}{2}}-e^{-i\frac{z}{2}}}{i}\right]_0^{\pi+2i} = \frac{e^{\frac{\pi}{2}i-1}-e^{1-\frac{\pi}{2}i}}{i}$

$= \frac{\frac{1}{e}\cdot i + e\cdot i}{i} = e+\frac{1}{e}$

(c) $\int_1^3 (z-2)^3 dz = [\frac{1}{4}(z-2)^4]_1^3 = 0$

Sect. 53. No. 1

(a) $f(z)$ is analytic in $\mathbb{C}\backslash\{-3\}$ and $-3 \notin \{z \mid |z|\leq 1\}$

Thus $f(z)$ is analytic in $\{z \mid |z|\leq 1\}$

(b) Obviously, $f(z)$ is analytic in $\{z \mid |z|\leq 1\}$.

(c) $f(z) = \frac{1}{z^2+2z+2} = \frac{1}{[z-(-1-i)][z-(-1+i)]}$

Since $-1-i$ and $-1+i$ are not belong to the set $\{z \mid |z|\leq 1\}$

we have $f(z)$ is analytic in $\{z \mid |z|\leq 1\}$

(d) $f(z) = \operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^z+e^{-z}}$

Suppose $e^z+e^{-z}=0$.

Then $e^{2z}=-1$.

i.e $e^{2x}e^{i2y} = -1$

Thus, $x=0$, $y=\frac{2k\pi+\pi}{2} = k\pi+\frac{\pi}{2}$ $k\in\mathbb{Z}$

Noted $|(0, k\pi+\frac{\pi}{2})| \geq \frac{\pi}{2} > 1$

Thus, $f(z)$ is analytic in $\{z \mid |z|\leq 1\}$

(e) $f(z) = \tan z = \frac{\sin z}{\cos z}$

Suppose $\cos z = 0$. Then $\frac{e^{iz}+e^{-iz}}{2} = 0$

$e^{2iz} = -1 \Rightarrow e^{2ix}e^{-2y} = -1 \Rightarrow y=0$, $x = \frac{\pi+2k\pi}{2}$ $k\in\mathbb{Z}$

Noted $|(k\pi+\frac{\pi}{2}, 0)| \geq \frac{\pi}{2} > 1$

Thus, $f(z)$ is analytic in $\{z \mid |z|\leq 1\}$

(f). $f(z) = \operatorname{Log}|z+2| = \ln|z+2| + i\operatorname{Arg}(z+2)$

Noted $z=-2 \notin \{z \mid |z|\leq 1\}$. Thus, $f(z)$ is well-defined and analytic in $\{z \mid |z|\leq 1\}$

Therefore, for (a)-(f), we have $\int_C f(z)dz = 0$ by Cauchy-Goursat Thm.